Solution to Problem Set 5

1. (a) Let $f(x) = \ln x$. Then we have $f'(x) = \frac{1}{x}$, so

$$f(0.98) \approx f(1) + f'(1)(-0.02)$$
$$\ln 0.98 \approx \ln 1 + (1/1)(-0.02) = -0.02.$$

(b) Let $f(x) = x^{10}$. Then we have $f'(x) = 10x^9$, so

$$f(1.013) \approx f(1) + f'(1)(0.013)$$
$$1.013^{10} \approx 1^{10} + 10(1^9)(0.013) = 1.13.$$

(c) Given $f(x) = \frac{2x}{1+x^2}$, we have $f'(x) = \frac{2(1+x^2)-(2x)(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$. So,

$$f(3.02) \approx f(3) + f'(3)(0.02) = \frac{2(3)}{1+3^2} + \frac{2-2(3^2)}{(1+3^2)^2}(0.02) = 0.6 - 0.0032 = 0.5968.$$

2. (a) Since $f'(x) = p(1+x)^{p-1}$, the linear approximation of f at 0 is given by

$$L(x) = f(0) + f'(0)(x-0) = (1+0)^p + p(1+0)^{p-1}(x-0) = px + 1.$$

(b) Since $g'(\theta) = \sec\theta\tan\theta$, the linear approximation of g at $\frac{\pi}{3}$ is given by

$$L(\theta) = g\left(\frac{\pi}{3}\right) + g'\left(\frac{\pi}{3}\right)\left(\theta - \frac{\pi}{3}\right) = \sec\frac{\pi}{3} + \sec\frac{\pi}{3}\tan\frac{\pi}{3}\left(\theta - \frac{\pi}{3}\right) = 2\sqrt{3}\theta + \frac{6-2\sqrt{3}\pi}{3}.$$

(c) Since $h'(t) = \frac{1}{t\ln t}$, the linear approximation of h at e is given by

$$L(t) = h(e) + h'(e)(t-e) = \ln\ln e + \frac{1}{e\ln e}(t-e) = \frac{1}{e}t - 1.$$

3. Denote the radius, the diameter and the volume of the sphere as r, D and V respectively. Now the maximum percentage error of the measurement of D is estimated to be 0.1%, so $\frac{\Delta D}{D} = 0.1\%$. The volume of the sphere is computed using the formula

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{\pi}{6}D^3.$$

Taking natural logarithms on both sides we get $\ln V = \ln\frac{\pi}{6} + 3\ln D$, and taking differentials we obtain

$$\frac{dV}{V} = 3\frac{dD}{D}.$$

This suggests the approximation

$$\frac{\Delta V}{V} \approx 3\frac{\Delta D}{D} = 3(0.1\%) = 0.3\%.$$

Therefore the maximum percentage error of the calculated volume will be about 0.3%.

4. Let $f(x) = x - 0.5\sin x - 1$. Then $f'(x) = 1 - 0.5\cos x$, and so the first iteration is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{1.5 - 0.5\sin 1.5 - 1}{1 - 0.5\cos 1.5} \approx 1.4987.$$

5. (a) Let $f(x) = x^3 + 2x - 4$. Then f is a continuous function. Since

$$f(1) = 1^3 + 2(1) - 4 = -1 < 0 \qquad \text{and} \qquad f(2) = 2^3 + 2(2) - 4 = 8 > 0,$$

the equation $f(x) = 0$ has **at least one** solution on $[1,2]$ by Intermediate Value Theorem. Moreover, since

$$f'(x) = 3x^2 + 2 > 0$$

for every $x \in \mathbb{R}$, it follows that f is strictly increasing on $\mathbb{R}$. This implies that f is one-to-one, and so the equation $f(x) = 0$ has **at most one** solution on $\mathbb{R}$. Therefore it has exactly one solution on $[1,2]$.

(b) In Newton's iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^3 + 2x_n - 4}{3{x_n}^2 + 2} = \frac{2{x_n}^3 + 4}{3{x_n}^2 + 2},$$

let's start with $x_0 = 1$. We have

$$x_1 = \frac{2{x_0}^3 + 4}{3{x_0}^2 + 2} = \frac{2(1)^3 + 4}{3(1)^2 + 2} = 1.200000$$

$$x_2 = \frac{2{x_1}^3 + 4}{3{x_1}^2 + 2} = \frac{2(1.200000)^3 + 4}{3(1.200000)^2 + 2} \approx 1.179747$$

$$x_3 = \frac{2{x_2}^3 + 4}{3{x_2}^2 + 2} \approx \frac{2(1.179747)^3 + 4}{3(1.179747)^2 + 2} \approx 1.179509$$

$$x_4 = \frac{2{x_3}^3 + 4}{3{x_3}^2 + 2} \approx \frac{2(1.179509)^3 + 4}{3(1.179509)^2 + 2} \approx 1.179509$$

Since x_3 and x_4 both give 1.1795 when corrected to 4 decimal places, we get the solution $x \approx 1.1795$.

6. Let $f(x) = \sin\ln x$. Then $f'(x) = (\cos\ln x)\left(\frac{1}{x}\right) = \frac{\cos\ln x}{x}$. Applying Newton's iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin\ln x_n}{\left(\dfrac{\cos\ln x_n}{x_n}\right)} = x_n(1 - \tan\ln x_n),$$

we have

$$\begin{aligned}
x_1 &= x_0(1 - \tan\ln x_0) = 27(1 - \tan\ln 27) \approx 22.802062,\\
x_2 &= x_1(1 - \tan\ln x_1) \approx 22.802062(1 - \tan\ln 22.802062) \approx 23.138227,\\
x_3 &= x_2(1 - \tan\ln x_2) \approx 23.138227(1 - \tan\ln 23.138227) \approx 23.140693,\\
x_4 &= x_3(1 - \tan\ln x_3) \approx 23.140693(1 - \tan\ln 23.140693) \approx 23.140693.
\end{aligned}$$

Since x_3 and x_4 both give 23.1407 when corrected to 4 decimal places, we obtain the solution

$$e^{\pi} \approx 23.1407.$$

Since $\sin\ln x = 0$ has solutions other than e^{π} (in fact for every integer n, $e^{n\pi}$ is a solution to this equation), the iterations may approach to another different solution if we had used another different initial trial x_0.

7. (a) We have

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} x e^{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{x}{e^{\frac{1}{x^2}}} = 0.$$

Now since $\lim_{x\to 0} h(x) = h(0)$, h is continuous at 0.

(b) We have

$$\lim_{x\to 0} \frac{h(x)-h(0)}{x-0} = \lim_{x\to 0} \frac{xe^{-\frac{1}{x^2}}-0}{x-0} = \lim_{x\to 0} e^{-\frac{1}{x^2}} = 0,$$

so h is differentiable at 0 and $h'(0) = 0$.

(c) For $x \neq 0$, we have

$$h'(x) = e^{-\frac{1}{x^2}} + xe^{-\frac{1}{x^2}} \frac{2}{x^3} = e^{-\frac{1}{x^2}}\left(1 + \frac{2}{x^2}\right).$$

Therefore

$$\lim_{x\to 0} h'(x) = \lim_{x\to 0} e^{-\frac{1}{x^2}}\left(1+\frac{2}{x^2}\right) = \lim_{x\to 0} \frac{1+\frac{2}{x^2}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{-\frac{4}{x^3}}{e^{\frac{1}{x^2}}\left(-\frac{2}{x^3}\right)} \qquad \text{(l'Hôpital's Rule)}$$

$$= \lim_{x\to 0} \frac{2}{e^{\frac{1}{x^2}}} = 0.$$

Now since $\lim_{x\to 0} h'(x) = h'(0)$, h' is continuous at 0.

8. f is differentiable at every $x \neq 0$, since it only consists of elementary functions and its denominator is never zero.

At 0, we have

$$f'_-(0) = \lim_{x\to 0^-} \frac{\frac{\sin 2x + \sin x}{e^x - 1} - (3e^0 - 0)}{x} = \lim_{x\to 0^-} \frac{\sin 2x + \sin x - 3(e^x - 1)}{x(e^x - 1)}$$

$$= \lim_{x\to 0^-} \frac{2\cos 2x + \cos x - 3e^x}{(e^x - 1) + xe^x} \qquad \text{(l'Hôpital's Rule)}$$

$$= \lim_{x\to 0^-} \frac{-4\sin 2x - \sin x - 3e^x}{e^x + (e^x + xe^x)} \qquad \text{(l'Hôpital's Rule)}$$

$$= \frac{-4(0) - 0 - 3e^0}{e^0 + (e^0 + 0e^0)} = -\frac{3}{2}$$

but

$$f'_+(0) = \lim_{x\to 0^+} \frac{(3e^x - x) - (3e^0 - 0)}{x} = \lim_{x\to 0^+} \frac{3e^x - x - 3}{x} = \lim_{x\to 0^+} \frac{3e^x - 1}{1} \qquad \text{(l'Hôpital's Rule)}$$

$$= \frac{3e^0 - 1}{1} = 2$$

Therefore f is not differentiable at 0.

9. (a)

$$\lim_{x\to 1}\frac{x-e^{x-1}}{(x-1)^2}=\lim_{x\to 1}\frac{1-e^{x-1}}{2(x-1)} \qquad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 1}\frac{-e^{x-1}}{2} \qquad \text{(l'Hôpital's Rule)}$$

$$=-\frac{1}{2}$$

(b) l'Hôpital's rule is not needed in this problem. In fact,

$$\lim_{x\to 0^+}\frac{\cot x}{\csc x}=\lim_{x\to 0^+}\cos x=1.$$

(c) Since $(\sin x)^x=e^{x\ln\sin x}$ for every $x\in(0,\pi)$, we first compute

$$\lim_{x\to 0^+}x\ln\sin x=\lim_{x\to 0^+}\frac{\ln\sin x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cos x}{-\frac{1}{x^2}} \qquad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 0^+}\frac{x}{\sin x}\cdot\lim_{x\to 0^+}-x\cos x=1\cdot 0=0.$$

Since the exponential function is continuous, we have

$$\lim_{x\to 0^+}(\sin x)^x=\lim_{x\to 0^+}e^{x\ln\sin x}=e^{\lim_{x\to 0^+}x\ln\sin x}=e^0=1.$$

(d) We have actually seen this problem before in Q9(g) of Problem Set 2! In fact,

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\left(\lim_{x\to 0}x\sin\frac{1}{x}\right)\left(\lim_{x\to 0}\frac{x}{\sin x}\right)=0\cdot 1=0.$$

(e) Since $(\ln x)^{\frac{1}{x-e}}=e^{\frac{1}{x-e}\ln\ln x}=e^{\frac{\ln\ln x}{x-e}}$ for every $x>e$, we first compute

$$\lim_{x\to e^+}\frac{\ln\ln x}{x-e}=\lim_{x\to e^+}\frac{\frac{1}{x\ln x}}{1} \qquad \text{(l'Hôpital's Rule)}$$

$$=\frac{1}{e\ln e}=\frac{1}{e}.$$

Since the exponential function is continuous, we have

$$\lim_{x\to e^+}(\ln x)^{\frac{1}{x-e}}=\lim_{x\to e^+}e^{\frac{\ln\ln x}{x-e}}=e^{\lim_{x\to e^+}\frac{\ln\ln x}{x-e}}=e^{\frac{1}{e}}.$$

(f) l'Hôpital's rule is not needed in this problem. In fact,

$$\lim_{x\to 0}\frac{x\tan x}{1-\sqrt{1-x^2}}=\lim_{x\to 0}\left(\frac{x\tan x}{1-\sqrt{1-x^2}}\cdot\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)=\lim_{x\to 0}\frac{\left(1+\sqrt{1-x^2}\right)\tan x}{x}$$

$$=\lim_{x\to 0}\left(\frac{1+\sqrt{1-x^2}}{\cos x}\cdot\frac{\sin x}{x}\right)=\frac{1+\sqrt{1-0^2}}{\cos 0}\cdot 1=2.$$

(g)

$$\lim_{x\to 1}\frac{1+\ln x-x^x}{1+\ln x-x}=\lim_{x\to 1}\frac{\frac{1}{x}-(x^x\ln x+x^x)}{\frac{1}{x}-1} \quad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 1}\frac{1-x^{x+1}\ln x-x^{x+1}}{1-x}$$

$$=\lim_{x\to 1}\frac{-\left[x^{x+1}\left(\ln x+\frac{x+1}{x}\right)\ln x+x^{x+1}\left(\frac{1}{x}\right)\right]-x^{x+1}\left(\ln x+\frac{x+1}{x}\right)}{-1} \quad \text{(l'Hôpital's Rule)}$$

$$=\frac{-[1(0+2)0+1]-1(1+2)}{-1}=3$$

(h) Note that

$$\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}=e^{\frac{1}{x^2}\ln\frac{\sin x}{x}}=e^{\frac{\ln|\sin x|-\ln|x|}{x^2}}$$

for every $x\in(-\pi,0)\cup(0,\pi)$, so we first compute $\lim_{x\to 0}\frac{\ln|\sin x|-\ln|x|}{x^2}$. Using the differentiation formula $\frac{d}{dx}\sin^2 x=2\sin x\cos x=\sin 2x$ when necessary, we obtain

$$\lim_{x\to 0}\frac{\ln|\sin x|-\ln|x|}{x^2}=\lim_{x\to 0}\frac{\frac{\cos x}{\sin x}-\frac{1}{x}}{2x} \quad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{x^2}-\frac{1}{\sin^2 x}}{2} \quad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{2x^2\sin^2 x}=\lim_{x\to 0}\frac{\sin 2x-2x}{4x\sin^2 x+2x^2\sin 2x} \quad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 0}\frac{2\cos 2x-2}{4\sin^2 x+8x\sin 2x+4x^2\cos 2x} \quad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 0}\frac{-4\sin 2x}{12\sin 2x+24x\cos 2x-8x^2\sin 2x} \quad \text{(l'Hôpital's Rule)}$$

$$=\lim_{x\to 0}\frac{-4}{12+12\frac{2x}{\sin 2x}\cos 2x-8x^2}=\frac{-4}{12+12\cdot 1\cdot 1-8\cdot 0}=-\frac{1}{6}.$$

Finally, since the exponential function is continuous, we have

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}=\lim_{x\to 0}e^{\frac{\ln|\sin x|-\ln|x|}{x^2}}=e^{\lim_{x\to 0}\frac{\ln|\sin x|-\ln|x|}{x^2}}=e^{-\frac{1}{6}}.$$

10. We have

$$\begin{aligned}
&\lim_{x\to a}\frac{f(x)-\left[f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2\right]}{(x-a)^2}\\
&=\lim_{x\to a}\frac{f'(x)-\left[f'(a)(1)+\frac{f''(a)}{2!}2(x-a)\right]}{2(x-a)} && \text{(l'Hôpital's rule)}\\
&=\lim_{x\to a}\frac{f'(x)-[f'(a)+f''(a)(x-a)]}{2(x-a)}\\
&=\lim_{x\to a}\frac{f''(x)-f''(a)}{2} && \text{(l'Hôpital's rule)}\\
&=\frac{f''(a)-f''(a)}{2}=0.
\end{aligned}$$

f'' is continuous at a

11. (a) The derivative of f is given by

$$f'(x)=\sqrt{3}\cos x-\sin x$$

for every $x\in(0,\pi)$. Now $f'(x)=0$ only if $\tan x=\sqrt{3}$, which happens only if $x=\frac{\pi}{3}$. So $\frac{\pi}{3}$ is the only interior critical number of f. Since

$$f(0)=\sqrt{3}\sin 0+\cos 0=1$$
$$f(\pi/3)=\sqrt{3}\sin(\pi/3)+\cos(\pi/3)=2$$
$$f(\pi)=\sqrt{3}\sin\pi+\cos\pi=-1,$$

we see that f has an absolute maximum point $\left(\frac{\pi}{3},2\right)$ and an absolute minimum point $(\pi,-1)$.

(b) Since 1 lies in the interval $[-1,3]$ while

$$\lim_{x\to 1^-}g(x)=\lim_{x\to 1^-}\frac{x^2}{x-1}=-\infty \qquad \text{and} \qquad \lim_{x\to 1^+}g(x)=\lim_{x\to 1^+}\frac{x^2}{x-1}=+\infty,$$

g does not have absolute extrema on $[-1,3]$.

(c) We can write

$$h(x)=\begin{cases}-(x-3)-(x+2) & \text{if } -4\le x<-2\\ -(x-3)+(x+2) & \text{if } -2\le x<3\\ (x-3)+(x+2) & \text{if } 3\le x\le 4\end{cases}=\begin{cases}1-2x & \text{if } -4\le x<-2\\ 5 & \text{if } -2\le x<3\\ 2x-1 & \text{if } 3\le x\le 4\end{cases}.$$

Easy computations give

$$h'(x)=\begin{cases}-2 & \text{if } -4<x<-2\\ 0 & \text{if } -2<x<3\\ 2 & \text{if } 3<x<4\end{cases}$$

and is undefined otherwise. So the interior critical numbers of h are all those numbers $c\in[-2,3]$. Since

$$h(-4)=1-2(-4)=9$$
$$h(4)=2(4)-1=7$$
$$h(c)=5 \qquad \text{for every } c\in[-2,3],$$

h has an absolute maximum point $(-4,9)$ and absolute minimum points $(c,5)$ for every $c\in[-2,3]$.